An Interpolation Theory Approach to H^∞ Controller Degree Bounds

D. J. N. Limebeer
Department of Electrical Engineering
Imperial College
Exhibition Road
London, England

and

B. D. O. Anderson
Department of Systems Engineering
Research School of Physical Sciences
Australian National University
Canberra, Australia

Submitted by M. Vidyasagar

ABSTRACT

We derive upper bounds for the McMillan degree of all H^∞-optimal controllers associated with design problems which may be embedded in a certain generalized regular configuration. Our analysis is confined to problems of the first kind, which are characterized by the assumption that both $P_{12}(s)$ and $P_{21}(s)$ are square but not necessarily of the same size. This paper, which uses interpolation theory, complements a previous paper which addresses the same problem through an approach based on approximation theory. We demonstrate that the interpolation theory approach is more direct and circumvents a number of the technical difficulties in the previous method; the final outcome is a much shorter proof. As a by-product, we achieve a new result on the degree of an optimal solution of the matrix Nevanlinna-Pick problem.

---

1. INTRODUCTION

In a recent paper [16] it was shown that there is a class of H^∞-optimal controllers with McMillan degree no greater than $n-1$ for all problems of the first kind, n being the McMillan degree of $P(s)$ in Figure 1. The proof in

52 Vanderbilt Ave., New York, NY 10017 0024-3795/88/$3.50

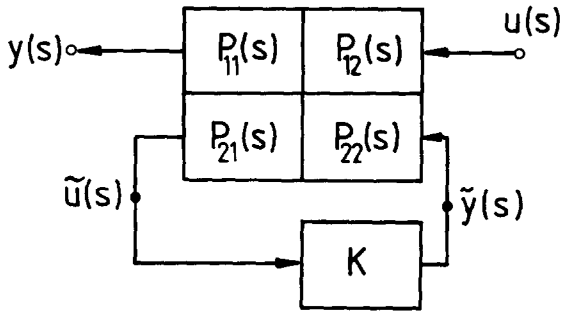

FIG. 1.

[16], which is based on Nehari approximation theory, is unfortunately long and intricate. With this in mind, a colleague of ours (M. C. Smith) remarked: "I wonder if one can get a shorter proof of the $n-1$ result using interpolation theory." In this paper we show that the answer to his question is yes. The main reason is that interpolation theory leads directly to a degree bound on all optimal closed loops, whereas bounding the degree of the closed loop using approximation theory is tortuous.

The Nevanlinna-Pick-Schur interpolation theory has already been used extensively in the study of single loop (SISO) H^∞ problems. Zames and Francis [25] use this theory to solve the minimum sensitivity problem. Their expression for the optimal closed loop [25, Equation (4.2)] in fact bounds the degree of the closed loop, the bound being the number of terms in the interpolating Blaschke product. Kimura [15] and Khargonekar and Tannenbaum [14] use interpolation theory to study optimal robustness problems. Tannenbaum has also analysed a "robustified" form of the strong stabilization problem using interpolation theory [21]. As with the work in [25], an optimal closed loop degree bound is implicit in this work. The situation is more complicated in the multivariable (MIMO) case.

One of the first solutions to the MIMO optimal sensitivity problem [4] uses matrix Nevanlinna theory [5]. Although it is easy to generalize the Chang-Pearson approach to general problems of the first kind, their use of the work in [5] disguises the McMillan degree of the closed loop. We obviate this problem by replacing the matrix Nevanlinna theory, à la [5], with a generalization of the so-called Nevanlinna-Pick tangent problem, which was first posed by M. G. Krein and studied by I. P. Fedčina [10–12]. This theory shows that all (matrix valued) closed loops can be characterized in terms of a Blaschke product of McMillan degree one factors. With this established, a closed loop degree bound follows without effort.

Our paper is laid out as follows: Section 2.1 defines the notation we need, and 2.2 gives a brief review of the Youla parametrization of all stabilizing compensators. The controller degree bound is derived in Section 3. It is established that the bound follows easily from classical Nevanlinna-Pick-Schur theory in the SISO case. By anticipating the solution of the Nevanlinna-Pick tangent problem, we extend the SISO analysis to the MIMO situation. Section 4.1 gives a precise statement of the generalized Nevanlinna-Pick tangent problem, and 4.2 considers some preliminary results on linear fractional maps. Conditions for the existence of solutions, and a characterization of all solutions to the tangent problem, are given in Section 4.3. The conclusions appear in Section 5.

2. NOTATION AND BACKGROUND THEORY

2.1. *Notation*

$\mathbb{R},\mathbb{C}$	Fields of real and complex numbers
$\mathbb{R}(s)$	Rational functions of s with real coefficients
$\mathbb{F}^{m\times l}$	$m\times l$ matrices with elements in $\mathbb{F}$ [$=\mathbb{R},\mathbb{C},\mathbb{R}(s)$, etc.]
A^*	Complex conjugate transpose of $A\in\mathbb{C}^{m\times l}$ (transpose if $A\in\mathbb{R}^{m\times l}$)
$\lambda(A)$	The set of eigenvalues of A
$A\geqslant 0,\ A>0$	A is positive semidefinite, positive definite
$A^{\#}$	Generalized inverse of A
RL^∞	Matrices in $\mathbb{R}^{m\times l}(s)$ which have no poles on the $j\omega$ axis (including ∞)
$\lVert\cdot\rVert_\infty$	L^∞ norm of matrices in RL^∞
RH^∞	Subspace of RL^∞ of matrices which have no poles in the open right half plane
$\bar{s}$	Complex conjugate of $s\in\mathbb{C}$
$\Rightarrow,\Leftarrow,\Leftrightarrow,\forall$	Implies, is implied by, if and only if, for all
$A(i,j)$	i,jth element of a matrix A
$\{\cdot\}_{j=1,l}^{i=1,m}$	A matrix whose rows are indexed from 1 to m and whose columns are indexed from 1 to l
$(A)_{ij}$	i,jth block of a matrix A
$\{(A)_{ij}\}_{j=1,l}^{i=1,m}$	A matrix whose block rows are indexed from 1 to m and whose block columns are indexed from 1 to l.

Associated with a transfer function matrix $G(s)\in\mathbb{R}^{m\times l}(s)$ of McMillan degree n is a state-space realization

$$G(s)=D+C(sI-A)^{-1}B, \tag{2.1a}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times l}$. To save space we often write (2.1a) as

$$G(s) = D + C(s-A)^{-1}B \tag{2.1b}$$

and make use of the alternative notation $G(s) \overset{s}{=} (A, B, C, D)$ or

$$G(s) \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2.2}$$

In the above notation, we have $G^*(-\bar{s}) \overset{s}{=} (-A^*, C^*, -B^*, D^*)$. If $G^*(-\bar{s})G(s) = I$, we say that $G(s)$ is all-pass. $G(s)$ is called stable (asymptotically stable) if it has no poles in the open (closed) right half plane. If $G(s)$ is stable and all-pass, we call it inner. If $G(s)$ is stable and $\|G(s)\|_\infty \leqslant 1$, we call it a matrix contraction. The space of matrix contractions is denoted by $\Theta^{m\times l}(s)$ or simply $\Theta(s)$ if an explicit reference to dimensions seems superfluous.

If $G(s) \overset{s}{=} (A, B, C, D)$, the system matrix corresponding to the given realization is defined as

$$\left[\begin{array}{c|c} sI - A & -B \\ \hline C & D \end{array}\right],$$

and the system zeros are defined as the points at which the system matrix loses normal rank. In the case that D is nonsingular, the system zeros are also given by $\lambda(A - BD^{-1}C)$. In general, the McMillan zeros of $G(s)$ are a subset of the system zeros of a realization of $G(s)$. The McMillan degree of $G(s)$ will be written as $\deg(G)$, and the set of poles (zeros) of $G(s)$ will be denoted {poles (zeros) of G}.

Let $P(s)$ be a partitioned matrix with a state-space realization given by

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]. \tag{2.3}$$

Then

$$P_{ij}(s) = C_i(s-A)^{-1}B_j + D_{ij}. \tag{2.4}$$

A linear fractional transformation of the partitioned matrix P and a matrix K

is defined as

$$F_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}, \tag{2.5}$$

where K is of dimension $l \times m$ if P_{22} has dimension $m \times l$. We associate with P a substitution matrix

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} P_{21} - P_{22}P_{12}^{-1}P_{11} & P_{22}P_{12}^{-1} \\ -P_{12}^{-1}P_{11} & P_{12}^{-1} \end{bmatrix}. \tag{2.6}$$

From Figure 1 we see that

$$y(s) = F_l(P,K)u(s)$$

and

$$\begin{bmatrix} \tilde{u}(s) \\ \tilde{y}(s) \end{bmatrix} = S \begin{bmatrix} u(s) \\ y(s) \end{bmatrix}. \tag{2.7}$$

We end this section by mentioning that if the hermitian matrix

$$H = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

is positive (semi-)definite, then so too is its Schur complement $C - B^*A^\# B$. This follows from the congruence transformation

$$\begin{bmatrix} I & -A^\# B \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} I & -A^\# B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C - B^*A^\# B \end{bmatrix}. \tag{2.8}$$

Also $\operatorname{rank}(H) = \operatorname{rank}(A) + \operatorname{rank}(C - B^*A^\# B)$. In our application A will always be nonsingular.

2.2. *Parametrization of All Stabilizing Controllers*

Let $P(s)$ in Figure 1 be given by

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}(s) \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \tag{2.9}$$

and suppose that (A, B_2, C_2) is stabilizable and detectable. Under these conditions $K(s)$ stabilizes the feedback system in Figure 1 if and only if it stabilizes $P_{22}(s)$. Further, such stabilizing compensators always exist [9, 19]. On the other hand, if (A, B_2, C_2) fails to be stabilizable and detectable, such compensators do not exist. Let

$$P_{22}(s) = N_r(s)D_r^{-1}(s) = D_l^{-1}(s)N_l(s) \tag{2.10}$$

be right and left rational coprime fractional factorizations of $P_{22}(s)$, and

$$\begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix}(s)\begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix}(s) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.11}$$

the corresponding Bezout identities. All the matrices in (2.11) belong to RH^∞, and the set of all compensators which stabilize $P_{22}(s)$, and thus also $P(s)$, is given by [7, 23]

$$K(s) = (U_l + D_rQ)(V_l - N_rQ)^{-1}(s) \tag{2.12}$$

$$= (V_r - QN_l)^{-1}(U_r + QD_l)(s), \tag{2.13}$$

where $Q(s) \in \mathrm{RH}^\infty$. It is easy to verify that

$$K(I + P_{22}K)^{-1}(s) = (U_l + D_rQ)D_l(s). \tag{2.14}$$

Hence

$$\begin{aligned} R(s) &= \left[P_{11} - P_{12}K(I + P_{22}K)^{-1}P_{21}\right](s) \\ &= \left[(P_{11} - P_{12}U_lD_lP_{21}) - (P_{12}D_r)Q(D_lP_{21})\right](s) \\ &= \left[T_{11} - T_{12}QT_{21}\right](s), \end{aligned} \tag{2.15}$$

where the $T_{ij}(s)$'s are defined in an obvious way. Equation (2.15) shows that $R(s)$ is parameterized linearly in $Q(s)$. Since $R(s) \in \mathrm{RH}^\infty$ if and only if $Q(s) \in \mathrm{RH}^\infty$, we would expect that T_{11}, T_{12}, and T_{22} all belong to RH^∞.

Since (A, B_2) is stabilizable, there exists a state feedback matrix F such that $A - B_2F$ is stable. Similarly, since (A, C_2) is detectable, there exists an output injection matrix H such that $A - HC_2$ is stable. Given any such pair of stabilizing matrices F and H, right and left coprime factorizations of $P_{22}(s)$ together with the solutions of the Bezout identities are given by [9, 17]

$$\begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix}(s) \overset{s}{=} \left[\begin{array}{c|cc} A - B_2F & B_2 & H \\ \hline -F & I & 0 \\ C_2 - D_{22}F & D_{22} & I \end{array}\right] \tag{2.16}$$

and

$$\begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix}(s) \overset{s}{=} \left[\begin{array}{c|cc} A - HC_2 & B_2 - HD_{22} & H \\ \hline F & I & 0 \\ -C_2 & -D_{22} & I \end{array}\right]. \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15) gives, after minor calculation [9, 16, 19],

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix}(s) \overset{s}{=} \left[\begin{array}{cc|cc} A - B_2F & B_2F & B_1 & B_2 \\ 0 & A - HC_2 & B_1 - HD_{21} & 0 \\ \hline C_1 - D_{12}F & D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right]. \tag{2.18}$$

Note that the $T_{ij}(s)$'s all belong to RH^∞, as expected. Contrary to approaches which are based on approximation theory, we do not require $T_{12}(s)$ and $T_{21}(s)$ to be inner [9, 16, 19].

REMARK 2.1. In the special case that $P_{22}(s)$ is stable, we may set $F = 0$ and $H = 0$. This gives $U_l = 0$, $U_r = 0$, $D_r = I$, $V_l = I$, $V_r = I$, and $N_r = N_l = P_{22}(s)$. Substituting these into (2.13) gives

$$Q(s) = K(I + P_{22}K)^{-1}(s),$$

which is the Q-parameterization of Zames [26].

3. A CONTROLLER DEGREE BOUND FOR H^∞ PROBLEMS OF THE FIRST KIND

In this section we present a simple argument, based on interpolation theory, which leads to McMillan degree bounds on all H^∞ optimal controllers (or suboptimal controllers) for problems of the first kind. We remind the reader that problems of the first kind have P_{12} and P_{21} square. It is also assumed that these blocks are nonsingular for all $s = j\omega$ (including the point at ∞).

Let $n = \deg(P)$, $r = \deg(R)$, and c be the number of internal pole-zero cancellations between $K(s)$ and $P(s)$ which occur as a result of closing the feedback loop in Figure 1. Clearly,

$$r = n + \deg(K) - c \quad \Leftrightarrow \quad \deg(K) = r + c - n. \tag{3.1}$$

To obtain an upper bound for $\deg(K)$, we proceed in two steps:

(1) We use interpolation theory to establish an upper bound r_b for the McMillan degree r of all optimal closed-loop transfer functions $R(s)$, and

(2) Theorem 3.1 (below) provides an upper bound c_b on c.

Given such bounds, we have

$$\deg(K) \leqslant r_b + c_b - n. \tag{3.2}$$

In the sequel we see that the open right half plane zeros of $P_{12}(s)$ and $P_{21}(s)$ play a central role. For this reason, the following notation is useful:

$$z_{12} := \text{number of right half plane zeros of } P_{12}(s)$$

and

$$z_{21} := \text{number of right half plane zeros of } P_{21}(s);$$

also

$$z = z_{12} + z_{21}.$$

3.1. *The Bound r_b*

In Section 2 we demonstrated that all internally stable closed loops may be parametrized as

$$R(s) = (T_{11} - T_{12}QT_{21})(s), \qquad Q \in \mathrm{RH}^\infty, \tag{3.3}$$

in which $T_{12}(s)$ and $T_{21}(s)$ are stable. Since $Q(s) \in \mathrm{RH}^\infty$, every right half plane zero of either $T_{12}(s)$ or $T_{21}(s)$ is also a zero of $T_{12}QT_{21}(s)$.

In the SISO case we therefore have that

$$r(s_i) = t_{11}(s_i) = b_i, \tag{3.4}$$

where s_i is any right half plane zero of either $t_{12}(s)$ or $t_{21}(s)$. The pairs $(s_i, b_i : i = 1,2,\ldots, z)$ are interpolation constraints to be satisfied by any $r(s)$ which corresponds to an internally stable closed loop. From the standard Nevanlinna-Pick-Schur interpolation theory [6, 22, 24] we know that there exists an interpolating function $r(s) \in \mathrm{RH}^\infty$, with $\|r(s)\|_\infty \leqslant \rho$, if and only if the hermitian Pick matrix

$$\Pi = \left\{ \frac{\rho^2 - b_i \bar{b}_j}{s_i + \bar{s}_j} \right\}_{j=1,z}^{i=1,z} \tag{3.5}$$

is positive semidefinite. It is also well known that [6, 22]:

(i) If $\Pi > 0$, there is a continuum of interpolating functions with

$$\deg(r) \leqslant z + \deg(u), \tag{3.6}$$

where $u(s)$ is an arbitrary stable contraction.

(ii) If $\Pi \geqslant 0$ is not definite, the interpolating function is unique with

$$\deg(r) = \mathrm{rank}(\Pi). \tag{3.7}$$

(iii) The minimum norm taken by any interpolating function is given by the solution of a generalized eigenvalue problem: Find the maximum value of ρ such that $\det \Pi$ is zero.

In the SISO case, therefore,

$$r_b = z - 1 \tag{3.8}$$

if $r(s)$ is optimal, or

$$r_b = z + \deg(u) \tag{3.9}$$

if $r(s)$ is suboptimal.

Multiinput, multioutput (MIMO) generalizations of these results will be given in Section 4. In fact all that changes is that (3.8) and (3.9) become

$$r_b = z + \deg(U) - 1 \tag{3.10}$$

and

$$r_b = z + \deg(U) \tag{3.11}$$

respectively. $U(s)$ is an arbitrary matrix contraction of specified dimensions. In particular, there exists an optimal interpolating matrix $R(s)$ for which $\deg(R) = z - 1$.

3.2. *The Bound c_b*

The key step behind establishing the bound c_b is to pin down the exact locations (in the frequency domain) of all the pole-zero cancellations which might occur as a result of closing the feedback loop in Figure 1. We then note that no cancellations may occur in the right half plane, since this would violate the proven internal stability of the closed loop.

In Theorem 3.1 we show that every unobservable mode of the closed loop in Figure 1 is due to a cancellation with a zero of $P_{12}(s)$, and every uncontrollable mode is due to a cancellation with a zero of $P_{21}(s)$. Given that this is true, we see that the number of cancellations c between $P(s)$ and $K(s)$ is bounded above by

$$\begin{aligned} c \leqslant &\{\text{number of left half plane zeros of } P_{12}(s)\} \\ &+ \{\text{number of left half plane zeros of } P_{21}(s)\} = c_b, \end{aligned}$$

or what is the same,

$$c \leqslant \{n - z_{12}\} + \{n - z_{21}\} = 2n - z = c_b. \tag{3.12}$$

THEOREM 3.1 [1, 16]. *Let*

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \tag{3.13}$$

in which $P_{12}(s) \in \mathbb{R}^{p_1 \times m_2}(s)$ with $p_1 \geqslant m_2$ and $P_{21}(s) \in \mathbb{R}^{p_2 \times m_1}(s)$ with $m_1 \geqslant p_2$. Suppose also that

$$K(s) \overset{s}{=} \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \tag{3.14}$$

is a minimal realization and that the well-posedness condition $\det(I - D_{22}\hat{D}) \neq 0$ *is satisfied. Then in the closed loop of Figure* 1,

(a) *every unobservable mode* (*from* y) *is a Smith zero of*

$$\left[\begin{array}{c|c} sI - A & B_2 \\ \hline C_1 & D_{12} \end{array}\right]; \tag{3.15}$$

(b) *every uncontrollable mode* (*from* u) *is a Smith zero of*

$$\left[\begin{array}{c|c} sI - A & B_1 \\ \hline C_2 & D_{21} \end{array}\right]. \tag{3.16}$$

Proof. The equations describing the closed loop of Figure 1 are

$$\dot{x} = Ax + B_1 u + B_2 \tilde{y},$$

$$y = C_1 x + D_{11} u + D_{12} \tilde{y},$$

$$\tilde{u} = C_2 x + D_{21} u + D_{22} \tilde{y},$$

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\tilde{u},$$

$$\tilde{y} = \hat{C}\hat{x} + \hat{D}\tilde{u}.$$

Eliminating the variables $\tilde{u}$ and $\tilde{y}$ leads to the following state-space model for the closed loop:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A + B_2\hat{D}MC_2 & B_2[I + \hat{D}MD_{22}]\hat{C} \\ \hat{B}MC_2 & \hat{A} + \hat{B}MD_{22}\hat{C} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_1 + B_2\hat{D}MD_{21} \\ \hat{B}MD_{21} \end{bmatrix} [u] \tag{3.17a}$$

in which

$$M := \left(I - D_{22}\hat{D}\right)^{-1} \tag{3.17b}$$

and

$$[y] = \left[C_1 + D_{12}\hat{D}MC_2 \quad D_{12}[I + \hat{D}MD_{22}]\hat{C}\right]\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \left[D_{11} + D_{12}\hat{D}MD_{21}\right][u]. \tag{3.18}$$

If s_0 is an unobservable mode of the closed loop state-space model (3.17) and (3.18), then there exists a vector $[w_1^* \;\; w_2^*]^* \neq 0$ such that

$$\left[\begin{array}{cc} s_0 I - A - B_2\hat{D}MC_2 & -B_2[I+\hat{D}MD_{22}]\hat{C} \\ -\hat{B}MC_2 & s_0 I - \hat{A} - \hat{B}MD_{22}\hat{C} \\ \hline C_1 + D_{12}\hat{D}MC_2 & D_{12}[I+\hat{D}MD_{22}]\hat{C} \end{array}\right]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0. \tag{3.19}$$

Defining

$$z_2 := \hat{D}MC_2 w_1 + [I + \hat{D}MD_{22}]\hat{C}w_2 \tag{3.20}$$

implies that

$$\left[\begin{array}{c|c} s_0 I - A & -B_2 \\ \hline C_1 & D_{12} \end{array}\right]\begin{bmatrix} w_1 \\ z_2 \end{bmatrix} = 0. \tag{3.21}$$

The proof of part (a) is concluded by establishing that $[w_1^* \;\; w_2^*] \neq 0 \Rightarrow [w_1^* \;\; z_2^*] \neq 0$. Suppose for contradiction that $[w_1^* \;\; z_2^*] = 0$. This implies that

$$\begin{aligned} & [I + \hat{D}MD_{22}]\hat{C}w_2 = 0 \\ \Leftrightarrow \quad & (I - \hat{D}D_{22})^{-1}\hat{C}w_2 = 0 \\ \Leftrightarrow \quad & \hat{C}w_2 = 0. \end{aligned} \tag{3.22}$$

We also have from (3.19) that

$$(s_0 I - \hat{A})w_2 = 0. \tag{3.23}$$

(3.22) and (3.23) taken together contradict the assumed minimality of (3.14), which proves condition (a). Part (b) may be established by a parallel sequence of arguments.

3.3. *The Controller Degree Bound*

The main theorem is proved by substituting (3.10), (3.11), and (3.12) into (3.2).

THEOREM 3.2. *For any H^∞-optimal control problem of the first kind, every H^∞-optimal controller satisfies*

$$\deg(K) \leqslant n + \deg(U) - 1, \tag{3.24}$$

and every ρ-suboptimal controller (i.e. $\|R(s)\|_\infty \leqslant \rho$) satisfies[1]

$$\deg(K) \leqslant n + \deg(U). \tag{3.25}$$

In (3.24) and (3.25) $U(s)$ is an arbitrary matrix conclusion of specified dimensions, which may have degree zero. In the SISO case the H^∞-optimal controller is unique and satisfies

$$\deg(K) \leqslant n - 1. \tag{3.26}$$

This bound is always also attainable in the MIMO case.

Note that for the scalar optimal control problem, the degree bound is proved in a fairly simple way. For the matrix case we have yet to establish the required Nevanlinna-Pick generalization.

4. THE NEVANLINNA-PICK TANGENT PROBLEM

We refer to the parametrization of all internally stable closed loops once more. That is

$$R(s) = T_{11} - T_{12}QT_{21}(s), \tag{4.1}$$

in which $T_{12}(s)$ and $T_{21}(s)$ are square with no imaginary axis zeros. Suppose in the MIMO case that s_i is any open right half plane zero of $T_{21}(s)$. Since this zero cannot be canceled, $T_{21}(s)$ must lose rank at this frequency, which implies that there exists a vector $a_i \neq 0$ such that

$$R(s_i)a_i = T_{11}(s_i)a_i = b_i, \qquad i = 1, 2, \ldots, z_{21}. \tag{4.2}$$

Similarly, if s_i are the right half plane zeros of $T_{12}(s)$, there exist vectors $a_i^* \neq 0$ such that

$$a_i^* R(s_i) = a_i^* T_{11}(s_i) = b_i^*, \qquad i = z_{21} + 1, \ldots, z. \tag{4.3}$$

[1] In the case of degenerate problems having no interpolation constraints, $\deg(R) = 0 \Rightarrow r_b = 0$, $c_b = 2n$ and consequently $\deg(K) \leqslant n + \deg(U)$.

Equations (4.2) and (4.3) taken together describe the MIMO interpolation constraints associated with all internally stable closed loops in the case of simple zeros. With this motivation, we now describe the general problem to be studied in the remainder of this section [10–12].

4.1. *The Problem Statement*

Given a (finite) sequence of points in the open right half plane

$$s_i, \qquad i = 1,2,\dots,n_r,$$

together with the sequences of vectors

$$a_i \in \mathbf{C}^p \text{ and } b_i \in \mathbf{C}^m, \qquad i = 1,2,\dots,n_r,$$

and a second sequence of points (also in the right half plane)

$$s_i, \qquad i = n_r + 1,\dots,n_r + n_l = n,$$

together with two further sequences of vectors

$$a_i \in \mathbf{C}^m \text{ and } b_i \in \mathbf{C}^p, \qquad i = n_r + 1,\dots,n:$$

(a) Find necessary and sufficient conditions for the existence of a stable interpolating matrix function $R(s) \in \mathbb{R}^{m\times p}(s)$ which satisfies both

$$R(s_i)a_i = b_i, \qquad i = 1,2,\dots,n_r, \tag{4.4}$$

and

$$a_i^* R(s_i) = b_i^*, \qquad i = n_r + 1,\dots,n. \tag{4.5}$$

(b) If the solution is not unique, characterize all interpolating matrices.

We alert the reader to the fact that we will not treat certain interpolation points with multiplicities. Specifically, if

$$R(s_i)a_{i1} = b_{i1}$$

and

$$R(s_i)a_{i2} = b_{i2},$$

then we assume that a_{i1} and a_{i2} are linearly independent. A similar remark applies to the case of $a_{i1}^{*}R(s_i) = b_{i1}^{*}$ and $a_{i2}^{*}R(s_i) = b_{i2}^{*}$. Finally, we do not allow

$$R(s_i)a_i = b_i$$

and

$$\hat{a}_i^{*}R(s_i) = \hat{b}_i^{*}.$$

The technical embellishments required to treat the general case of zeros with multiplicities may be found in [12]. We have decided not to go into these details because the main result of this paper has already been proven in the most general case using approximation theory [16].

It is well known that inner matrices and Blaschke products (products of J-unitary matrices[1]) play a fundamental role in the study of interpolation problems. Before we begin the main attack on the Nevanlinna-Pick tangent problem, we give some preliminary results on (contractive) linear fractional transformations and J-unitary matrices; this will be the subject of the next section. Since most of these results are known, albeit in a different form, our treatment will be terse (see [8, 18] for further details).

4.2. *Some Preliminary Results on Linear Fractional Transformations*

LEMMA 4.1 (A contractive property of linear fractional maps). *Suppose*

$$H(s) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}(s)$$

is all-pass, that $H_{12}(s)$ and $H_{21}(s)$ are square and nonsingular almost everywhere, and that

$$R(s) = \big(H_{11} + H_{12}U(I - H_{22}U)^{-1}H_{21}\big)(s). \tag{4.6}$$

Then

(i) *We have*

$$I - U^{*}(j\omega)U(j\omega) \geqslant 0 \quad \Leftrightarrow \quad I - R^{*}(j\omega)R(j\omega) \geqslant 0. \tag{4.7}$$

[1] J-unitary matrices satisfy $A(s)JA^{*}(s) = J$, where $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

(ii) *We have*

$$I - U^*(j\omega)U(j\omega) = 0 \quad \Leftrightarrow \quad I - R^*(j\omega)R(j\omega) = 0. \tag{4.8}$$

(iii) *If* $H_{12}^*(j\omega)H_{12}(j\omega) > 0\ \forall\omega$, *and* $H(s)$ *is asymptotically stable with* $U(s)$ *a stable matrix contraction, then* $R(s)$ *is stable and contractive too.*

(iv) *The substitution matrix associated with* $H(s)$ *satisfies* $S^*(-\bar{s})JS(s) = J$, *i.e., it is J-unitary.*

Proof. We will not show the frequency dependence of the various matrices explicitly. The assumed all-pass character of $H(s)$ gives

$$H_{11}^*H_{11} + H_{21}^*H_{21} = I, \tag{4.9}$$

$$H_{12}^*H_{12} + H_{22}^*H_{22} = I, \tag{4.10}$$

$$H_{11}^*H_{12} + H_{21}^*H_{22} = 0, \tag{4.11}$$

and a simple calculation based on these equations will establish that

$$I - R^*R = H_{21}^*(I - U^*H_{22}^*)^{-1}\{I - U^*U\}(I - H_{22}U)^{-1}H_{21}.$$

(i) and (ii) are thus established.

From (4.10) we get $H_{22}^*H_{22} = I - H_{12}^*H_{12} < I\ \forall\omega \Rightarrow \|H_{22}(s)\|_\infty < 1$. Since $\|U(s)\|_\infty \leqslant 1$, and since $U(s)$ and $H(s)$ are stable, $(I - H_{22}U)^{-1}$ must also be stable by the small gain theorem. Consequently $R(s)$ is stable as required.

By definition

$$S = \begin{bmatrix} H_{21} - H_{22}H_{12}^{-1}H_{11} & H_{22}H_{12}^{-1} \\ -H_{12}^{-1}H_{11} & H_{12}^{-1} \end{bmatrix},$$

and a direct computation based on (4.9) to (4.11) will establish its J-unitary character. This proves (iv) and completes the proof. ■

The Nevanlinna algorithm makes extensive use of elementary linear fractional maps. As we will show, these maps characterize all matrix functions which satisfy a single interpolation constraint [10].

LEMMA 4.2 (Properties of elementary linear fractional maps). *Suppose* s_1 *is a complex number in the open right half plane and that* a *and* b *are* *complex vectors which satisfy* $(a^*a - b^*b) > 0$. *If*

$$H(s) \stackrel{s}{=} \left[\begin{array}{c|cc} -\bar{s}_1 + \phi b^*b & -a^* & b^* \\ \hline \phi b & 0 & I \\ -\phi a & I & 0 \end{array}\right], \tag{4.12}$$

in which

$$\phi = -\frac{s_1 + \bar{s}_1}{a^*a - b^*b}, \tag{4.13}$$

then:

(i) $H(s)$ *is inner.*
(ii) *The substitution matrix associated with* $H(s)$ *is*

$$S(s) = \left[\begin{array}{c|cc} -\bar{s}_1 & a^* & -b^* \\ \hline \phi a & I & 0 \\ \phi b & 0 & I \end{array}\right] \tag{4.14}$$

which is J-unitary.
(iii) *If* $R(s) = [H_{11} + H_{12}U(I - H_{22}U)^{-1}H_{21}](s)$, *then* $R(s)$ *is a stable contraction and* $R(s_1)a = b$ $\forall U(s) \in \Theta(s)$.

Proof. (i) and (ii) follow by an easy calculation, which we omit. Since $\mathrm{Re}(\bar{s}_1 - \phi b^*b) > 0$, $H(s)$ is stable. $H_{12}(s)$ has its only zero at $-s_1$, so $H_{12}^*(j\omega)H_{12}(j\omega) > 0$ $\forall\omega$, so $R(s)$ is stable and contractive by Lemma 4.1. Finally,

$$H_{11}(s_1) = \frac{-\phi ba^*}{s_1 + \bar{s}_1 - \phi b^*b} = \frac{ba^*}{a^*a} \quad\Rightarrow\quad H_{11}(s_1)a = b$$

and

$$H_{21}(s_1) = \left[I + \frac{\phi aa^*}{s_1 + \bar{s}_1 - \phi b^*b}\right] = \left[I - \frac{aa^*}{a^*a}\right] \quad\Rightarrow\quad H_{21}(s_1)a = 0,$$

which establishes the interpolating property of $R(s)$ and completes the proof. ■

We will also make use of the state-space characterization of the J-unitary property.

LEMMA 4.3. *Suppose that $G(s)$ is square with a minimal realization $G(s) \overset{s}{:=} (A, B, C, D)$. Then $G(s)$ is J-unitary if and only if there exists a $Q = Q^*$ such that*

$$A^*Q + QA + C^*JC = 0, \tag{4.15}$$

$$D^*JC + B^*Q = 0, \tag{4.16}$$

$$JD^*J = D^{-1}. \tag{4.17}$$

Proof. All one need do is replace $G^*(-\bar{s})G(s) = I$ with $G^*(-\bar{s})JG(s) = J$ and repeat the arguments given in Theorem 5.1 of Glover [13]. ∎

We conclude this section with a result which shows that the linear fractional map

$$R(s) = F_l(H(s), U(s)), \qquad U(s) \in \Theta(s), \tag{4.18}$$

generates all matrix functions $R(s) \in \Theta(s)$ which satisfy the interpolation constraint $R(s_1)a = b$.

LEMMA 4.4. *Suppose that $H(s)$ is defined as in* (4.12). *Then:*

(i) *If $\tilde{R}(s)$ is any matrix contraction satisfying $\tilde{R}(s_1)a = b$ with $a^*a - b^*b > 0$, then there exists a $\tilde{U}(s) \in \Theta(s)$ such that*

$$\tilde{R}(s) = F_l(H(s), \tilde{U}(s)). \tag{4.19}$$

(ii) *If $(a^*a - b^*b) = 0$,*

$$R(s) = F_l(H, U(s)) \tag{4.20}$$

generates all interpolating matrix functions satisfying $R(s_1)a = b$. The constant matrix H is completely determined by a and b given in (4.27) *below.*

Proof. Solving for $\tilde{U}(s)$ from (4.19) gives

$$\tilde{U}(s) = [S_{22}\tilde{R} + S_{21}][S_{12}\tilde{R} + S_{11}]^{-1}(s), \tag{4.21}$$

and we need to establish that $\tilde{U}(s) \in \Theta(s)$. By invoking the J-unitary character of $S(s)$ it is easily verified that

$$I - \tilde{U}^*(s)\tilde{U}(s) = [S_{12}R + S_{11}]^{-*}[I - \tilde{R}^*\tilde{R}][S_{12}R + S_{11}]^{-1}(s). \tag{4.22}$$

This establishes that $\|\tilde{U}(s)\|_\infty \leqslant 1$. Clearly,

$$\tilde{U}(s) = [S_{22}\tilde{R} + S_{21}][I + S_{11}^{-1}S_{12}\tilde{R}]^{-1}S_{11}^{-1}. \tag{4.23}$$

It is trivial to show that S_{11}^{-1} is stable and nonsingular $\forall j\omega$. The J-unitary property of $S(s)$ ensures that $S_{11}S_{11}^* - S_{12}S_{12}^* = I$, whence $S_{11}^{-1}S_{12}$ is stable and strictly contractive. Since $\tilde{R}$ is contractive also, the small gain theorem ensures that $(S_{11}^{-1}S_{12}\tilde{R} + I)^{-1}$ is stable and consequently so too is $\tilde{U}(s)$. This completes the proof of (i).

If $a^*a - b^*b = 0$, we note that there exist unitary matrices Z and V such that

$$Za = \begin{bmatrix} \sigma \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad Vb = \begin{bmatrix} \sigma \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4.24}$$

and we may write

$$[V_1^* \quad V_2^*]\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & U(s) & \\ 0 & & & \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} a = b, \qquad U(s) \in \Theta(s). \tag{4.25}$$

Thus

$$F_l(H, U(s))a = b, \tag{4.26}$$

where

$$H = \begin{bmatrix} V_1{}^*Z_1 & V_2{}^* \\ Z_2 & 0 \end{bmatrix}. \tag{4.27}$$

If $\tilde{R}(s)$ is any matrix contraction satisfying $\tilde{R}(s_1)a = b$, we have

$$V\tilde{R}(s_1)Z^*Za = Vb. \tag{4.28}$$

Invoking (4.24), the fact that $V\tilde{R}(s)Z^* \in \Theta(s)$, and a standard argument based on the maximum modulus principle [22] gives

$$V\tilde{R}(s)Z^* = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & V_2\tilde{R}(s)Z_2^* & \\ 0 & & & \end{bmatrix}, \tag{4.29}$$

so that

$$\begin{aligned} \tilde{R}(s) &= \begin{bmatrix} V_1{}^* & V_2{}^* \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & V_2\tilde{R}(s)Z_2^* & \\ 0 & & & \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \\ &= F_l\left(H, V_2\tilde{R}(s)Z_2^*\right). \end{aligned} \tag{4.30}$$

This shows that in the case $a^*a - b^*b = 0$, (4.26) generates all interpolating matrix functions which satisfy $R(s_1)a = b$ when $R(s) \in \Theta(s)$. The proof is thus complete. ∎

4.3. *Solution of the Nevanlinna-Pick Tangent Problem*

In this section we establish necessary and sufficient conditions for the existence of a solution to the tangent problem. We will also prove that for any stable $R(s)$ which satisfies the constraints given in (4.4) and (4.5),

$$\deg(R) \leqslant n + \deg(U), \tag{4.31a}$$

where $U(s)$ is any element of $\Theta^{m\times p}(s)$. If $R(s)$ is any minimum norm interpolating function, then[2]

$$\deg(R) \leqslant n + \deg(U) - 1; \tag{4.31b}$$

again, $U(s)$ is a free parameter in $\Theta(s)$.

THEOREM 4.1. *There exists a stable $m\times p$ matrix function $R(s)$ with $\|R(s)\|_\infty \leqslant \rho$ satisfying the interpolation constraints* (4.4) *and* (4.5) *if and only if the matrix*

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \tag{4.32}$$

is positive semidefinite, where

$$\Pi_{11} = \left\{ \frac{\rho^2 a_i^* a_k - b_i^* b_k}{\bar{s}_i + s_k} \right\}_{k=1,n_r}^{i=1,n_r}, \qquad \Pi_{12} = \left\{ \frac{\rho a_i^* b_k - \rho b_i^* a_k}{\bar{s}_i - \bar{s}_k} \right\}_{k=n_r+1,n}^{i=1,n_r}$$

and

$$\Pi_{12}^* = \left\{ \frac{\rho b_i^* a_k - \rho a_i^* b_k}{s_k - s_i} \right\}_{k=1,n_r}^{i=n_r+1,n}, \qquad \Pi_{22} = \left\{ \frac{\rho^2 a_i^* a_k - b_i^* b_k}{\bar{s}_k + s_i} \right\}_{k=n_r+1,n}^{i=n_r+1,n}.$$

Further, if $\Pi > 0$,

$$\deg(R) \leqslant n + \deg(U), \tag{4.33}$$

and if $\Pi \geqslant 0$,

$$\deg(R) \leqslant n + \deg(U) - 1, \tag{4.34}$$

where $U(s)$ is a free parameter of appropriate dimension belonging to $\Theta(s)$.

---

[2]Again, we assume that there is at least one interpolation constraint.

REMARK 4.1. As we will now show, the calculation of the minimum value of ρ for which an interpolating matrix function exists is a hermitian eigenvalue problem. We begin by expanding Π as

$$\Pi = \rho^2 A_0 + \rho A_1 + A_2, \tag{4.35}$$

in which A_0, A_1 and A_2 may be easily identified from (4.32). Further, $A_0 = A_0^* > 0$, $A_1 = A_1^*$, and $A_2 = A_2^* \leqslant 0$; the hermitian nature of the three matrices in (4.35) is obvious, while the definiteness of A_0 and A_2 follow from a simple Lyapunov equation argument.

Next, we make the following series of observations:

(i) We have

$$\begin{bmatrix} I & 0 \\ \rho A_0 & I \end{bmatrix} H \begin{bmatrix} I & \rho A_0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} -A_0^{-1} & 0 \\ 0 & \Pi \end{bmatrix}, \tag{4.36}$$

in which

$$H = \rho \begin{bmatrix} 0 & I \\ I & A_1 \end{bmatrix} - \begin{bmatrix} A_0^{-1} & 0 \\ 0 & -A_2 \end{bmatrix}. \tag{4.37}$$

(ii) H is a hermitian pencil and consequently has real eigenvalues.
(iii) H is singular if and only if Π is singular.

From the positive definiteness of A_0, it follows that $\Pi \geqslant 0$ if and only if $\rho \geqslant \lambda_{\max}(H)$, where $\lambda_{\max}(H)$ is the maximum ρ in (4.37) for which H is singular [moreover, $\Pi > 0$ if $\rho > \lambda_{\max}(H)$]. In other words, the minimum norm of any interpolating matrix function is given by $\lambda_{\max}(H)$.

REMARK 4.2. In the case that $\Pi \geqslant 0$ (rather than $\Pi > 0$), the interpolating matrix function may be unique. Conditions for uniqueness appear at the end of the proof of sufficiency.

REMARK 4.3. We will assume from now on that the interpolation constraints have been normalized by replacing the b_i's with $\rho^{-1} b_i = \hat{b}_i$, $i = 1, 2, \ldots, n$. Once an interpolating function—call it $\hat{R}(s)$—has been found for the $\hat{b}_i$'s $R(s) = \rho \hat{R}(s)$ will be an interpolating function for the b_i's.

The proof of necessity requires a preliminary result which we will now prove.

LEMMA 4.5. *Let $Z(s)$ be the $n \times n$ Laplace transform of a causal impulse response $z(t)$ mapping inputs in L_n^2 into outputs in L_n^2 via*

$$y(t) = z(t) * u(t) = \int_{-\infty}^{t} z(t-\tau)u(\tau)\,d\tau \tag{4.38}$$

($u(\cdot)$ and $y(\cdot)$ may be complex). Consider an input defined by

$$u(t) = \sum_{i=1}^{n_r} a_i e^{s_i t}, \qquad t \leqslant 0, \quad \operatorname{Re}(s_i) > 0, \tag{4.39a}$$

$$u(t) = \sum_{i=n_r+1}^{n} a_i e^{-s_i t}, \qquad t > 0, \quad \operatorname{Re}(s_i) > 0. \tag{4.39b}$$

Then

$$\operatorname{Re}\int_{-\infty}^{+\infty} u^*(t)y(t)\,dt = \tfrac{1}{2}[a_1^*, a_2^*, \ldots, a_n^*]\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \tag{4.40}$$

with $A \in \mathbf{C}^{n_r \times n_r}$, $B \in \mathbf{C}^{n_r \times n_l}$, and $C \in \mathbf{C}^{n_l \times n_l}$ block matrices, where

$$(A)_{ij} = \frac{Z^*(s_i) + Z(s_j)}{\bar{s}_i + s_j}, \qquad 1 \leqslant i, j \leqslant n_r, \tag{4.41a}$$

$$(B)_{ij} = \frac{Z^*(\bar{s}_j) - Z^*(s_i)}{\bar{s}_i - s_j}, \qquad 1 \leqslant i \leqslant n_r, \quad n_r + 1 \leqslant j \leqslant n, \tag{4.41b}$$

$$(C)_{ij} = \frac{Z(\bar{s}_i) + Z^*(\bar{s}_j)}{\bar{s}_i + s_j}, \qquad n_r + 1 \leqslant i,\ j \leqslant n. \tag{4.41c}$$

Proof. We observe first that for $t \in (-\infty, 0]$,

$$y(t) = \sum_{i=1}^{n_r} Z(s_i) a_i e^{s_i t}. \tag{4.42}$$

Hence

$$\operatorname{Re}\int_{-\infty}^{+\infty} u^*(t) y(t)\, dt$$

$$= \operatorname{Re}\left\{ \sum_{i=1}^{n_r} \int_{-\infty}^{0} a_i^* e^{\bar{s}_i t} \sum_{j=1}^{n_r} Z(s_j) a_j e^{s_j t}\, dt \right\}$$

$$+ \operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \int_0^{\infty} a_i^* e^{-\bar{s}_i t}\, dt \int_{-\infty}^{0} z(t-\tau) \sum_{j=1}^{n_r} a_j e^{s_j \tau}\, d\tau \right\}$$

$$+ \operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \int_0^{\infty} a_i^* e^{-\bar{s}_i t}\, dt \int_0^{t} z(t-\tau) \sum_{j=n_r+1}^{n} a_j e^{-s_j \tau}\, d\tau \right\}$$

$$= \operatorname{Re} \sum_{i=1}^{n_r} \sum_{j=1}^{n_r} \frac{a_i^* Z(s_j) a_j}{\bar{s}_i + s_j} + I_2 + I_3$$

$$= \frac{1}{2} \sum_{i=1}^{n_r} \sum_{j=1}^{n_r} \frac{a_i^* \left[Z^*(s_i) + Z(s_j) \right] a_j}{\bar{s}_i + s_j} + I_2 + I_3. \tag{4.43}$$

We evaluate I_2 next:

$$I_2 = \operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \int_0^{\infty} a_i^* e^{-\bar{s}_i t}\, dt \int_{-\infty}^{0} z(t-\tau) \sum_{j=1}^{n_r} a_j e^{s_j \tau}\, d\tau \right\}.$$

Substituting $\xi = t - \tau$, $\eta = t + \tau$ and noting that

$$\frac{\partial(\xi, \eta)}{\partial(t, \tau)} = 2 \tag{4.44}$$

gives

$$I_2 = \tfrac{1}{2}\operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \sum_{j=1}^{n_r} \int_0^\infty a_i^* \, d\xi \int_{-\xi}^{\xi} e^{-\bar{s}_i(\xi+\eta)/2} z(\xi) e^{s_j(\eta-\xi)/2} a_j \, d\eta \right\}$$

$$= \tfrac{1}{2}\operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \sum_{j=1}^{n_r} \int_0^\infty a_i^* e^{-(\bar{s}_i+s_j)\xi/2} Z(\xi) \, d\xi \int_{-\xi}^{\xi} e^{(s_j-\bar{s}_i)\eta/2} \, d\eta \, a_j \right\}$$

$$= \operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \sum_{j=1}^{n_r} \int_0^\infty a_i^* e^{-(\bar{s}_i+s_j)\xi/2} Z(\xi) \frac{e^{(s_j-\bar{s}_i)\xi/2} - e^{-(s_j-\bar{s}_i)\xi/2}}{s_j - \bar{s}_i} a_j \, d\xi \right\}$$

$$= \operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \sum_{j=1}^{n_r} \frac{a_i^*\left[Z(\bar{s}_i) - Z(s_j)\right] a_j}{s_j - \bar{s}_i} \right.$$

$$= \frac{1}{2} \sum_{i=n_r+1}^{n} \sum_{j=1}^{n_r} \frac{a_i^*\left[Z(\bar{s}_i) - Z(s_j)\right] a_j}{s_j - \bar{s}_j}$$

$$+ \frac{1}{2} \sum_{i=n_r+1}^{n} \sum_{j=1}^{n_r} \frac{a_j^*\left[Z^*(\bar{s}_i) - Z^*(s_j)\right] a_i}{\bar{s}_j - s_i}. \tag{4.45}$$

I_3 is given by

$$I_3 = \operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \sum_{j=n_r+1}^{n} \int_0^\infty a_i^* e^{-\bar{s}_i t} \, dt \int_0^t Z(t-\tau) a_j e^{-s_j \tau} \, d\tau \right\}.$$

Setting $t - \tau = \mu$ gives

$$I_3 = \operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \sum_{j=n_r+1}^{n} \int_0^\infty a_i^* e^{-(\bar{s}_i+s_j)\tau} \, d\tau \int_0^\infty Z(\mu) e^{-\bar{s}_i \mu} \, d\mu \, a_j \right\}$$

$$= \operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \sum_{j=n_r+1}^{n} \frac{a_i^* Z(\bar{s}_i) a_j}{\bar{s}_i + s_j} \right\}$$

$$= \operatorname{Re}\left\{ \sum_{i=n_r+1}^{n} \sum_{j=n_r+1}^{n} \frac{a_i^*\left[Z(\bar{s}_i) + Z^*(\bar{s}_j)\right] a_j}{2(\bar{s}_i + s_j)} \right\}, \tag{4.46}$$

which completes the proof. ∎

The condition

$$\operatorname{Re}\int_{-\infty}^{+\infty} u^*(t)y(t)\,dt \geqslant 0 \tag{4.47}$$

for $u(t) \in L_n^2$ is a condition for passivity. This combined with the fact that $z(\cdot)$ is a convolution operator mapping L_n^2 into L_n^2 implies that $Z(s)$ is positive real.

What we have thus shown is that if $Z(s)$ is positive real, and $s_1,\dots,s_{n_r}, s_{n_r+1},\dots,s_n$ are arbitrary points in the open right half plane, then the matrix

$$\mathscr{X} = \begin{bmatrix} \mathscr{X}_{11} & \mathscr{X}_{12} \\ \mathscr{X}_{12}^* & \mathscr{X}_{22} \end{bmatrix}, \tag{4.48}$$

where

$$\mathscr{X}_{11} = \left\{ \left(\frac{Z^*(s_i) + Z(s_j)}{\bar{s}_i + s_j} \right)_{i,j} \right\}_{j=1,n_r}^{i=1,n_r},$$

$$\mathscr{X}_{12} = \left\{ \left(\frac{Z^*(\bar{s}_j) - Z^*(s_i)}{\bar{s}_i - s_j} \right)_{i,j} \right\}_{j=n_r+1,n}^{i=1,n_r},$$

$$\mathscr{X}_{22} = \left\{ \left(\frac{Z(\bar{s}_i) + Z^*(\bar{s}_j)}{\bar{s}_i + s_j} \right)_{i,j} \right\}_{j=n_r+1,n}^{i=n_r+1,n},$$

is necessarily positive semidefinite.

Proof of necessity. Our purpose here is to show that if an interpolating matrix valued function exists, $\Pi \geqslant 0$.

In his original paper, Pick proved necessity using Cauchy's integral formula. In another paper on scalar interpolation with positive real functions, Youla and Saito [24] proved necessity using a Riesz-Herglotz representation of positive real functions. Youla and Saito also give a pretty circuit theoretic argument based on energy ideas. In the matrix case, Delsarte, Genin, and Kamp [5] also use Riesz-Herglotz theory to prove necessity.

The idea of our proof is to replace the positive real matrix in Lemma 4.5 with a matrix expression involving a bounded real matrix $S(s)$ only. This then allows us to show that a matrix like (4.48), but which is in terms of $S(s)$, is also positive semidefinite. The proof is then simply completed by substituting the interpolation constraints.

It is well known from passive circuit theory [2] that if $Z(s)$ is positive real,

$$S(s) = [Z(s) - I][Z(s) + I]^{-1} \tag{4.49a}$$

is bounded real. Invoking the inverse relation,

$$Z(s) = [I + S(s)][I - S(s)]^{-1}, \tag{4.49b}$$

gives

$$\begin{aligned} Z^*(s_i) + Z(s_j) &= [I - S^*(s_i)]^{-1} \\ &\quad \times \{[I + S^*(s_i)][I - S(s_j)] + [I - S^*(s_i)][I + S(s_j)]\} \\ &\quad \times [I - S(s_j)]^{-1} \\ &= 2[I - S^*(s_i)]^{-1}\{I - S^*(s_i)S(s_j)\}[I - S(s_j)]^{-1}. \end{aligned} \tag{4.50a}$$

Similarly,

$$Z^*(\bar{s}_j) - Z^*(s_i) = 2[I - S^*(s_i)]^{-1}\{S^*(\bar{s}_j) - S^*(s_i)\}[I - S^*(\bar{s}_j)]^{-1}, \tag{4.50b}$$

and finally,

$$Z(\bar{s}_i) + Z^*(\bar{s}_j) = 2[I - S(\bar{s}_i)]^{-1}\{I - S(\bar{s}_i)S^*(\bar{s}_j)\}[I - S(\bar{s}_j)]^{-1}. \tag{4.50c}$$

Consequently,

$$\mathscr{X} = 2\Delta\begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12}^* & \zeta_{22} \end{bmatrix}\Delta^*, \tag{4.51}$$

where

$$\zeta_{11} = \left\{ \frac{I - S^*(s_i)S(s_j)}{\bar{s}_i + s_j} \right\}_{j=1,n_r}^{i=1,n_r},$$

$$\zeta_{12} = \left\{ \frac{S^*(\bar{s}_j) - S^*(s_i)}{\bar{s}_i - s_j} \right\}_{j=n_r+1,n}^{i=1,n_r},$$

$$\zeta_{22} = \left\{ \frac{I - S(\bar{s}_i)S^*(\bar{s}_j)}{\bar{s}_i + s_j} \right\}_{j=n_r+1,n}^{i=n_r+1,n},$$

and

$$\Delta = \operatorname{diag}\left\{ [I - S^*(s_i)]^{-1}, [I - S(\bar{s}_i)]^{-1} \right\}.$$

We therefore conclude that if $S(s)$ is bounded real, then

$$\zeta = \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{12}^* & \zeta_{22} \end{bmatrix} \geqslant 0. \tag{4.52}$$

If the bounded real interpolating matrix function (which we assume exists) is square, we set $S(s) = R(s)$ and recall also that

$$S(s_i)a_i = b_j, \qquad i = 1,2,\ldots,n_r, \tag{4.53a}$$

and

$$a_i^* S(\bar{s}_i) = b_i^*, \qquad i = n_r + 1,\ldots,n. \tag{4.53b}$$

Postmultiplying by $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ and premultiplying by $\operatorname{diag}(a_1^*, a_2^*, \ldots, a_n^*)$ immediately establishes that $\Pi \geqslant 0$.

If the interpolating matrix function is nonsquare, we note that $\|R(s)\|_\infty \leqslant 1$ implies that the $(m+p)\times(m+p)$ matrix

$$S(s) := \begin{bmatrix} R(s) & 0 \\ 0 & 0 \end{bmatrix} \tag{4.54}$$

is bounded real. Introducing the augmented vectors

$$\tilde{a}_i = \begin{bmatrix} a_j \\ 0 \end{bmatrix} \in \mathbf{C}^{m+p} \quad \text{and} \quad \tilde{b}_i = \begin{bmatrix} b_j \\ 0 \end{bmatrix} \in \mathbf{C}^{m+p}, \tag{4.55}$$

which satisfy

$$S(s_i)\tilde{a}_i = \tilde{b}_i, \qquad i = 1,2,\dots,n_r, \tag{4.56a}$$

and

$$\tilde{a}_i^* S(\bar{s}_i) = \tilde{b}_i^*, \qquad i = n_r + 1,\dots,n, \tag{4.56b}$$

allows the nonsquare problem to be treated as if it were square. This completes the proof of necessity. ■

Proof of sufficiency. The proof of sufficiency is inductive. At each step of the algorithm an elementary linear fractional map is used to reduce the number of interpolation constraints by one. We show that in the iterative construction of all interpolating matrix functions, the sequence of interpolation problems have associated with them a sequence of Pick-like matrices which have monotonically decreasing dimensions. Each of these matrices may be related to Π by a Schur complement argument. The algorithm is terminated by solving an interpolation problem with a single constraint in the case $\Pi > 0$, and at most two (matrix valued) constraints in the case $\Pi \geqslant 0$.

Case 1: $\Pi > 0$. The case of $n = 1$ can be solved immediately. We remark also that a left constraint can be transformed into a right constraint by simply writing

$$W^*(s_1)a_1 = b_1. \tag{4.57}$$

If $a_1^* a_1 - b_1^* b_1 > 0$, it follows from the properties of elementary linear fractional maps that

$$F_l(H(s_1), U(s))a_1 = b_1 \qquad \forall U(s) \in \Theta(s). \tag{4.58}$$

Further, all interpolating matrix functions will be generated as $U(s)$ ranges over $\Theta(s)$. $H(s)$ is defined in (4.12) and (4.13).

When tackling the general problem we will deal with the right constraints first, followed by the left constraints. Obviously, the algorithm must work for any ordering of the interpolation constraints; the present enumeration is employed solely for clarity of exposition. In the first step of the algorithm we eliminate the constraint (s_1, a_1, b_1). Since $\Pi > 0$, we have $a_1^* a_1 - b_1^* b_1 > 0$, and

$$H(s) \overset{s}{:=} \left[\begin{array}{c|cc} -\bar{s}_1 + \phi_1 b_1^* b_1 & -a_1^* & b_1^* \\ \hline \phi_1 b_1 & 0 & I \\ -\phi_1 a_1 & I & 0 \end{array}\right] \tag{4.59}$$

is defined, in which

$$\phi_1 = -\frac{s_1 + \bar{s}_1}{a_1^* a_1 - b_1^* b_1}. \tag{4.60}$$

Further,

$$F_l(H(s), U(s)) a_1 = b_1 \qquad \forall U(s) \in \Theta(s). \tag{4.61}$$

The remaining $n-1$ constraints are now fed down to the next step of the algorithm by making use of the substitution matrices associated with the diagrams in Figure 2.

The idea is to transform the problem of interpolating $(s_j, a_j, b_j : j = 1, 2, \ldots, n)$ into one of the interpolating $(s_j, \tilde{a}_j, \tilde{b}_j : j = 2, 3, \ldots, n)$. Direct calculation shows that

$$\begin{bmatrix} \tilde{a}_j \\ \tilde{b}_j \end{bmatrix} = S_R(s_j) \begin{bmatrix} a_j \\ b_j \end{bmatrix}, \qquad j = 2, 3, \ldots, n_r, \tag{4.62}$$

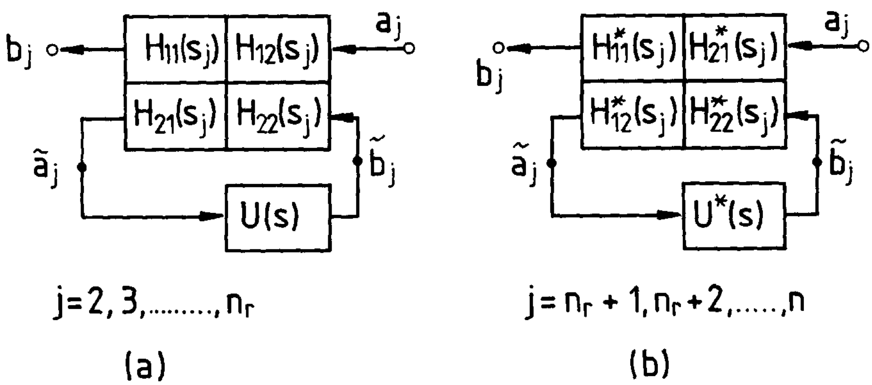

FIG. 2.

and

$$\begin{bmatrix} \tilde{b}_j \\ \tilde{a}_j \end{bmatrix} = S_L(\bar{s}_j)\begin{bmatrix} b_j \\ a_j \end{bmatrix}, \qquad j = n_r + 1, \ldots, n, \tag{4.63}$$

in which

$$S_R(s) \overset{s}{:=} \left[\begin{array}{c|cc} -\bar{s}_1 & a_1^* & -b_1^* \\ \hline \phi_1 a_1 & I & 0 \\ \phi_1 b_1 & 0 & I \end{array}\right] =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{4.64}$$

and

$$S_L(s) \overset{s}{:=} \left[\begin{array}{c|c} -A & -B \\ \hline C & D \end{array}\right]. \tag{4.65}$$

The subscripts L and R distinguish between the substitution matrices for left and right constraints. Having found expressions for the $n-1$ constraints $(s_j, \tilde{a}_j, \tilde{b}_j\colon j = 2,3,\ldots,n)$, we now calculate their associated Pick matrix and link it to Π. Clearly

$$\tilde{a}_j^*\tilde{a}_k - \tilde{b}_j^*\tilde{b}_k = \begin{bmatrix} a_j^* & b_j^* \end{bmatrix} S_R^*(s_j) J S_R(s_k) \begin{bmatrix} a_k \\ b_k \end{bmatrix}, \qquad j,k = 2,3,\ldots,n_r. \tag{4.66}$$

Since $S_R(s)$ is J-unitary, we have by Lemma 4.3 that

$$\begin{aligned} S_R^*(s_j) J S_R(s_k) &= D^*JD + D^*JC(s_k + A)^{-1}B + B^*(\bar{s}_j - A^*)^{-1}C^*JD \\ &\quad + B^*(\bar{s}_j - A^*)^{-1}C^*JC(s_k - A)^{-1}B \\ &= J - B^*Q(s_k - A)^{-1}B - B^*(\bar{s}_j - A^*)^{-1}QB \\ &\quad + B^*(\bar{s}_j - A^*)^{-1}C^*JC(s_k - A)^{-1}B \\ &= J + B^*(\bar{s}_j - A^*)^{-1} \\ &\quad \times \left\{C^*JC - Q(s_k - A) - (\bar{s}_j - A^*)Q\right\}(s_k - A)^{-1}B \\ &= J - (s_k + \bar{s}_j)B^*(\bar{s}_j - A^*)^{-1}Q(s_k - A)^{-1}B, \end{aligned} \tag{4.67}$$

where Q solves the J-unitary equation (4.15). Substituting (4.64) into (4.15) gives

$$Q=\frac{s_1+\bar{s}_1}{a_1^*a_1-b_1^*b_1}. \tag{4.68}$$

Substituting (4.64), (4.67), and (4.68) into (4.66) yields after a minor manipulation

$$\left\{\frac{\tilde{a}_j^*\tilde{a}_k-\tilde{b}_j^*\tilde{b}_k}{\bar{s}_j+s_k}=\frac{a_j^*a_k-b_j^*b_k}{\bar{s}_j+s_k}-\frac{a_j^*a_1-b_j^*b_1}{\bar{s}_j+s_1}\times\frac{s_1+\bar{s}_1}{a_1^*a_1-b_1^*b_1}\times\frac{a_1^*a_k-b_1^*b_k}{\bar{s}_1+s_k}\right\}_{k=2,n_r}^{j=2,n_r}. \tag{4.69}$$

In the same way we have that

$$\tilde{a}_j^*\tilde{b}_k-\tilde{b}_j^*\tilde{a}_k=\begin{bmatrix}a_j^* & b_j^*\end{bmatrix}S_R^*(s_j)JS_L(\bar{s}_k)\begin{bmatrix}b_k\\ a_k\end{bmatrix},$$
$$j=2,3,\ldots,n_r,\quad k=n_r+1,\ldots,n, \tag{4.70}$$

$$\tilde{b}_j^*\tilde{a}_k-\tilde{a}_j^*\tilde{b}_k=\begin{bmatrix}b_j^* & a_j^*\end{bmatrix}S_L^*(\bar{s}_j)JS_R(s_k)\begin{bmatrix}a_k\\ b_k\end{bmatrix},$$
$$j=n_r+1,\ldots,n,\quad k=2,3,\ldots,n_r, \tag{4.71}$$

$$\tilde{a}_j^*\tilde{a}_k-\tilde{b}_j^*\tilde{b}_k=-\begin{bmatrix}b_j^* & a_j^*\end{bmatrix}S_L^*(\bar{s}_j)JS_L(\bar{s}_k)\begin{bmatrix}b_k\\ a_k\end{bmatrix},$$
$$j,k=n_r+1,\ldots,n. \tag{4.72}$$

After elementary computations (which we will spare the reader) we get that (4.70) $\Rightarrow$

$$\left\{\frac{\tilde{a}_j^*\tilde{b}_k-\tilde{b}_j^*\tilde{a}_k}{\bar{s}_j-\bar{s}_k}=\frac{a_j^*b_k-b_j^*a_k}{\bar{s}_j-\bar{s}_k}-\frac{a_j^*a_1-b_j^*b_1}{\bar{s}_j+s_1}\times\frac{s_1+\bar{s}_1}{a_1^*a_1-b_1^*b_1}\times\frac{a_1^*b_k-b_1^*a_k}{\bar{s}_1-\bar{s}_k}\right\}_{k=n_r+1,n}^{j=2,n_r}, \tag{4.73}$$

(4.71) $\Rightarrow$

$$\left\{ \frac{\tilde{b}_j^* \tilde{a}_k - \tilde{a}_j^* \tilde{b}_k}{s_k - s_j} = \frac{b_j^* a_k - a_j^* b_k}{s_k - s_j} - \frac{b_j^* a_1 - a_j^* b_1}{s_1 - s_j} \right.$$

$$\left. \times \frac{s_1 + \bar{s}_1}{a_1^* a_1 - b_1^* b_1} \times \frac{a_1^* a_k - b_1^* b_k}{\bar{s}_1 + s_k} \right\}_{k = 2, n_r}^{j = n_r + 1, n}, \tag{4.74}$$

and (4.72) $\Rightarrow$

$$\left\{ \frac{\tilde{a}_j^* \tilde{a}_k - \tilde{b}_j^* \tilde{b}_k}{s_j + \bar{s}_k} = \frac{a_j^* a_k - b_j^* b_k}{\bar{s}_k + s_j} - \frac{b_j^* a_1 - a_j^* b_1}{s_1 - s_j} \right.$$

$$\left. \times \frac{s_1 + \bar{s}_1}{a_1^* a_1 - b_1^* b_1} \times \frac{a_1^* b_k - b_1^* a_k}{\bar{s}_1 - \bar{s}_k} \right\}_{k = n_r + 1, n}^{j = n_r + 1, n}. \tag{4.75}$$

It is now easy to see that the left hand sides of (4.69), (4.73), (4.74), and (4.75) taken together form the $(n-1)\times(n-1)$ Pick matrix associated with $(s_j, \tilde{a}_j, \tilde{b}_j \colon j = 2, 3, n)$. We observe next, and this is most interesting, that the right hand sides of these same equations are a Schur complement of Π. Suppose that partitioning Π below the first row and to the right of the first column gives

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{12}^* & \tilde{\Pi}_{22} \end{bmatrix}. \tag{4.76}$$

Then $\tilde{\Pi}_{22} - \pi_{12}^* \pi_{11}^{-1} \pi_{12}$ is the Schur complement we seek. Note that $\Pi > 0 \Rightarrow \tilde{\Pi}_{22} - \pi_{12}^* \pi_{11}^{-1} \pi_{12} > 0$. Consequently we have established that the problem of n constraints may be reduced to a problem with $n-1$ constraints, and the corresponding $(n-1)\times(n-1)$ Pick matrix is again positive definite. A repetition of these arguments reduces the original problem to one containing a single constraint (which we have already solved). Since all interpolation functions are generated by a chain of n elementary linear fractional maps terminated by $U(s) \in \Theta(s)$, and since each linear fractional map has McMillan degree one, we obtain (4.33) as required. As is well known, the chain of elementary linear fractional maps may be replaced with a Blaschke product with n degree one factors. This completes the proof of sufficiency in the case $\Pi > 0$.

Case 2: $\Pi \geqslant 0$. When considering the case $\Pi \geqslant 0$, we suppose that $\operatorname{rank}(\Pi) = r < n$. We will assume also that the interpolation constraints have been ordered so that the first r successive principal minors of Π are nonzero. Under these assumptions the first r steps of the Nevanlinna algorithm may be carried out as before, and simple rank considerations show that the $(n-r)\times(n-r)$ Pick matrix associated with the remaining $n-r$ interpolation constraints vanishes identically. After an appropriate ordering of the remaining constraints, the $(n-r)\times(n-r)$ Pick matrix becomes

$$\begin{bmatrix} A_R^* A_R - B_R^* B_R & A_R^* B_L - B_R^* A_L \\ B_L^* A_R - A_L^* B_R & A_L^* A_L - B_L^* B_L \end{bmatrix} = 0. \tag{4.77}$$

$A_R \in \mathbb{C}^{p\times\alpha_R}$ and $B_R \in \mathbb{C}^{m\times\alpha_R}$ are matrices whose columns are the right constraints, while $B_L \in \mathbb{C}^{p\times\alpha_L}$ and $A_L \in \mathbb{C}^{m\times\alpha_L}$ are matrices whose columns are left constraints; α_R is the number of right constraints, and α_L is the number of left constraints.

The $(1,1)$ block of (4.77) establishes the existence of unitary matrices $Q \in \mathbb{C}^{p\times p}$ and $Z \in \mathbb{C}^{m\times m}$ such that

$$QA_R = \begin{bmatrix} E \\ 0 \end{bmatrix} \quad \text{and} \quad ZB_R = \begin{bmatrix} E \\ 0 \end{bmatrix}, \tag{4.78}$$

where E has full row rank ρ_R. Consequently

$$\begin{bmatrix} I_{\rho_R} & 0 \\ 0 & U(s) \end{bmatrix} QA_R = ZB_R, \tag{4.79}$$

in which $U(s) \in \Theta^{(m-\rho_R)\times(p-\rho_R)}(s)$. Thus

$$[Z_1^* \quad Z_2^*] \begin{bmatrix} I_{\rho_R} & 0 \\ 0 & U(s) \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} A_R = B_R, \tag{4.80}$$

or equivalently,

$$[Z_1^* Q_1 + Z_2^* U(s) Q_2] A_R = B_R \qquad \forall U(s) \in \Theta(s), \tag{4.81}$$

which may be expressed in terms of the contractive linear fractional map

$$F_l(H_R, U(s)) A_R = B_R, \tag{4.82}$$

in which

$$H_R = \begin{bmatrix} Z_1^*Q_1 & Z_2^* \\ Q_2 & 0 \end{bmatrix}. \tag{4.83}$$

A simple calculation which is based on Figure 2(b) and which exploits the unitary character of Q and Z shows that the substitution matrix S_L is given by

$$S_L = \begin{bmatrix} Z_2 & 0 \\ 0 & Q_2 \end{bmatrix}. \tag{4.84}$$

Thus

$$\begin{bmatrix} \tilde{A}_L \\ \tilde{B}_L \end{bmatrix} \begin{bmatrix} Z_2 A_L \\ Q_2 B_L \end{bmatrix}, \tag{4.85}$$

and hence

$$\begin{aligned} \tilde{A}_L^* \tilde{A}_L - \tilde{B}_L^* \tilde{B}_L &= A_L^* Z_2^* Z_2 A_L - B_L^* Q_2^* Q_2 B_L \\ &= A_L^* A_L - B_L^* B_L - A_L^* Z_1^* Z_1 A_L + B_L^* Q_1^* Q_1 B_L \\ &= B_L^* Q_1^* Q_1 B_L - A_L^* Z_1^* Z_1 A_L \end{aligned}$$

[by the (2,2) block of (4.77)].

The (2,1) block of (4.77) gives

$$B_L^* Q^* Q A_R - A_L^* Z^* Z B_R = 0$$

$$\Rightarrow \qquad B_L^* \begin{bmatrix} Q_1^* & Q_2^* \end{bmatrix} \begin{bmatrix} E \\ 0 \end{bmatrix} - A_L^* \begin{bmatrix} Z_1^* & Z_2^* \end{bmatrix} \begin{bmatrix} E \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow \qquad (B_L^* Q_1^* - A_L^* Z_1^*) E = 0$$

$$\Rightarrow \qquad B_L^* Q_1^* - A_L^* Z_1^* = 0$$

(since E has full row rank)

$$\Rightarrow \qquad \tilde{A}_L^* \tilde{A}_L - \tilde{B}_L^* \tilde{B}_L = 0 \tag{4.86}$$

Consequently, there exist unitary matrices $N \in \mathbf{C}^{(m-p_R)\times(m-p_R)}$ and $M \in$

$\mathbf{C}^{(p-\rho_R)\times(p-\rho_R)}$ such that

$$N\tilde{A}_L = \begin{bmatrix} \tilde{E} \\ 0 \end{bmatrix} \quad \text{and} \quad M\tilde{B}_L = \begin{bmatrix} \tilde{E} \\ 0 \end{bmatrix}, \tag{4.87}$$

in which $\tilde{E}$ has rank ρ_L, say. Thus

$$M^* \begin{bmatrix} I_{\rho_L} & 0 \\ 0 & U(s) \end{bmatrix} N\tilde{A}_L = \tilde{B}_L \tag{4.88}$$

with $U(s) \in \Theta^{(p-\rho)\times(m-\rho)}$, $\rho := \rho_R + \rho_L$. An obvious partitioning of M and N now yields

$$[M_1^* N_1 + M_2^* U(s) N_2]\tilde{A}_L = \tilde{B}_L. \tag{4.89}$$

Substituting (4.89) into (4.85) gives

$$(Q_2^* M_1^* N_1 Z_2 + Q_2^* M_2^* U N_2 Z_2) A_L = B_L.$$

Combining this with (4.82) shows that the class of functions

$$F_l(H, U(s)), \tag{4.90}$$

in which $U(s) \in \Theta^{(p-\rho)\times(m-\rho)}(s)$ and

$$H = \begin{bmatrix} Z_1^* Q_1 + Z_2^* N_1^* M_1 Q_2 & Z_2^* N_2^* \\ M_2 Q_2 & 0 \end{bmatrix}, \tag{4.91}$$

satisfy all the interpolation constraints in (4.77). The linear frctional map in (4.90) [which has McMillan degree $\deg(U)$] terminates the algorithm for $\Pi \geqslant 0$. An argument similar to that given at the end of Section 4.2 establishes that (4.90) generates all interpolating functions which satisfy the constraints given in (4.77).

Note: If $\rho \geqslant \min(p, m)$, the matrix valued interpolation function is unique.

Since the interpolating function is constructed from r elementary linear fractional maps (each with McMillan degree one), we have that

$$\deg(R) \leqslant r + \deg(U) \leqslant n + \deg(U) - 1, \tag{4.92}$$

which completes the proof. ∎

REMARK 4.4 (Boundary interpolation). There are several instances in H^∞ control problems where it is necessary to do boundary interpolation. In the optimal sensitivity problem, for example, there will be boundary interpolation constraints whenever the plant has either poles or zeros on the imaginary axis (this includes the point at infinity). As has already been pointed out [14, 20], boundary interpolation may be accomplished with the aid of a simple bilinear transform. Suppose

$$\tilde{s} = \frac{s+\epsilon}{1+\epsilon s}, \qquad \epsilon > 0; \tag{4.93}$$

then this conformal map transforms the *closed* right half s-plane onto a circle centered on the positive real axis with diameter $[\epsilon, 1/\epsilon]$ in the $\tilde{s}$-plane. To do boundary interpolation, one simply transforms the original problem in s into a new problem in $\tilde{s}$. The transformed problem requires no boundary interpolation and may thus be solved using the techniques already described. Once an interpolating matrix $\tilde{R}(\tilde{s})$ for the transformed problem has been found, it is converted into $R(s)$ for the original problem using (4.93).

Suppose $q \leqslant n_r$ of the n_r right constraints are in the open right half plane, while the remainder lie on the $j\omega$-axis. Similarly, we assume that $l \leqslant n_l$ of the left constraints are in the open right half plane with the remainder on the $j\omega$-axis. If

$$\begin{aligned}\Xi = \operatorname{diag}\Big\{ & I_q, \big[2\operatorname{Re}(\tilde{s}_{q+1})\big]^{1/2}, \dots, \big[2\operatorname{Re}(\tilde{s}_{n_r})\big]^{1/2}, \\ & I_l, \big[2\operatorname{Re}(\tilde{s}_{n_r+l+1})\big]^{1/2}, \dots, \big[2\operatorname{Re}(\tilde{s}_n)\big]^{1/2}\Big\},\end{aligned}$$

then the Pick matrix $\Pi(\tilde{s})$ (in the transformed variable $\tilde{s}$) will be positive definite if and only if

$$\hat{\Pi}(\tilde{s}) = \Xi \Pi(\tilde{s}) \Xi^*$$

is positive definite for any $\epsilon > 0$. A lower bound for the norm of the interpolating matrix function may be obtained by considering

$$\lim_{\epsilon \to 0} \hat{\Pi}(\tilde{s}) = \begin{bmatrix} (1,1) & 0 & (1,3) & 0 \\ 0 & (2,2) & 0 & 0 \\ (3,1) & 0 & (3,3) & 0 \\ 0 & 0 & 0 & (4,4) \end{bmatrix},$$

in which

$$\begin{bmatrix} (1,1) & (1,3) \\ (3,1) & (3,3) \end{bmatrix}$$

is the Pick matrix corresponding to the open right half plane points in the original s variable. The (2,2) and (4,4) account for the boundary points and have the form

$$\operatorname{diag}\left(\rho^2 - \|b_i\|_2^2\right),$$

where we assume that $\|a_i\|_2^2 = 1$. Consequently, the lower bound we seek (which may not be attainable) is given by

$$\mu = \max\left\{ \lambda_{\max}\begin{bmatrix} (1,1) & (1,3) \\ (3,1) & (3,3) \end{bmatrix}, \|b_{q+1}\|_2, \dots, \|b_{n_r}\|_2, \|b_{n_r+l+1}\|_2, \|b_n\|_2 \right\}.$$

5. CONCLUSIONS

The purpose of this paper was to obtain an H^∞-optimal controller degree bound for problems of the first kind using interpolation theory. This complements the analysis in [16], which is based on approximation theory. Apart from being of independent theoretical interest, the interpolation theory proof is shorter. In the SISO case the result is almost immediate if one assumes the classical Nevanlinna-Pick-Schur theory. In the MIMO case it was necessary to generalize the Nevanlinna-Pick tangent theory of Fedčina. Despite the need for this generalization, it is the authors' opinion that the interpolation theory approach is pedagogically appealing. In this general case of interpolation constraints with multiplicities, there seems to be little to choose between the approximation theory and interpolation theory approaches; they are both complicated.

It is a pleasure to thank Michael Green for several helpful suggestions made during the course of this work. The present research was carried out while the first author was a Visiting Fellow with the Department of System Engineering, The Australian National University, Canberra, Australia.

REFERENCES

1 B. D. O. Anderson and A. Linnemann, Control of decentralized systems with distributed controller complexity, in *Proceedings 24th IEEE CDC*, 1985, pp. 1468–1472.

2 B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis. A Modern Approach*, Prentice-Hall, 1972.

3 S. Barnett, *Matrices in Control Theory*, Van Nostrand Reinhold, London, 1971.

4 B. C. Chang and B. Pearson, Optimal disturbance rejection in linear multivariable systems, *IEEE Trans. Automat. Control* AC-29:880–887 (1984).

5 Ph. Delsarte, Y. Genin, and Y. Kamp, The Nevanlinna-Pick problem for matrix valued functions, *SIAM J. Appl. Math.* 36:47–61 (1979).

6 Ph. Delsarte, Y. Genin, and Y. Kamp, On the role of the Nevanlinna-Pick problem in circuit and system theory, *Circuit Theory Appl.* 9:177–187 (1981).

7 C. A. Desoer, R. W. Liu, J. Murray, and R. Saeks, Feedback system design: The fractional representation approach to analysis and synthesis, *IEEE Trans. Automat. Control* AC-25:399–412 (1980).

8 P. Dewilde and H. Dym, Lossless chain scattering matrices and optimum linear prediction: The vector case, *Circuit Theory Appl.* 9:135–175 (1981).

9 J. C. Doyle et al., *Advances in Multivariable Control*, ONR/Honeywell Workshop, 1984.

10 I. P. Fedčina, A criterion for the solubility of the Nevanlinna-Pick tangent problem (in Russian), *Mat. Issled.* 7, vyp. 4(26):213–227 (1972).

11 I. P. Fedčina, A description of the solutions of the Nevanlinna-Pick tangent problem (in Russian), *Akad. Nauk Armyan. SSR Dokl.* 60:37–42 (1975).

12 I. P. Fedčina, The Nevanlinna-Pick tangent problem with multiple points (in Russian), *Akad. Nauk Armyan. SSR Dokl.* 61:214–218 (1975).

13 K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds, *Internat. J. Control* 39:1115–1193 (1984).

14 P. P. Khargonekar and A. Tannenbaum, Non-Euclidean metrics and robust stabilization of systems with parameter uncertainty, *IEEE Trans. Automat. Control* AC-30:1005–1013 (1985).

15 H. Kimura, Robust stabilization for a class of transfer functions, *IEEE Trans. Automat. Control* AC-29:788–793 (1984).

16 D. J. N. Limebeer and Y. S. Hung, An analysis of the pole-zero cancellations in H^∞ optimal control problems of the first kind, *SIAM J. Control Optim.*, 25:1457–1493 (1987).

17 C. N. Nett, C. A. Jacobson, and M. J. Balas, A connection between state-space and doubly coprime fractional representations, *IEEE Trans. Automat. Control* AC-29:831–832 (1984).

18 V. P. Potapov, The multiplicative structure of *J*-contractive matrix functions, *Trans. Amer. Math. Soc.* 15:131–243 (1960).

19 M. G. Safonov, E. A. Jonckheere, M. Verma, and D. J. N. Limebeer, Synthesis of positive real multivariable feedback systems, *Internat. J. Control*, 45:817–842 (1987).

20 M. G. Safonov, Imaginary axis zeros in multivarible H^∞ optimal control, in

Proceedings of the NATO Workshop on Modelling, Robustness and Sensitivity Reduction in Control Systems, Groningen, The Netherlands, 1–5 December 1986.

21 A. Tannenbaum, Feedback stabilization of linear dynamical plants with uncertainty in the gain factor, *Internat. J. Control* 32:1–16 (1980).

22 J. L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, A. M. S. Colloquium Publications, 20, 1965.

23 D. C. Youla, H. Jabr, and J. J. Bongiorno, Modern Wiener-Hopf design of optimal controllers, Part II: The multivariable case, *IEEE Trans. Automat. Control* AC-21:319–338 (1976).

24 D. C. Youla and M. Saito, Interpolation with positive real functions, *J. Franklin Inst.* 284:77–108 (1976).

25 G. Zames and B. A. Francis, Feedback, minimax sensitivity, and optimal robustness, *IEEE Trans. Automat. Control* AC-28:585–600 (1983).

26 G. Zames, Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses, *IEEE Trans. Automat. Control* AC-26:301–320 (1981).

Received March 1987; final manuscript accepted 1 July 1987